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Perceptuomotor Integration in Mathematics Learning

Playing Mathematical Instruments:

Emerging Perceptuomotor Integration With an Interactive Mathematics Exhibit

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Abstract

Research in experimental and developmental psychology, cognitive science, and neuroscience, suggests that tool fluency depends on the merging of perceptual and motor aspects of its use, an achievement we call *perceptuomotor integration*. We investigate the development of perceptuomotor integration and its role in mathematical thinking and learning. Just as expertise in playing a piano relies on the interanimation of finger movements and perceived sounds, we argue that mathematical expertise involves the systematic interpenetration of perceptual and motor aspects of playing *mathematical instruments*. Through 2 microethnographic case studies of visitors who engaged with an interactive mathematics exhibit in a science museum, we explore the real-time emergence of perceptuomotor integration and the ways in which it supports mathematical imagination.

Key words: Functions; Knowledge; Learning; Learning theories; Philosophical issues; Reasoning; Technology (general)

Contemporary research on the nature of mathematical thinking and learning highlights that mathematical development includes the skillful appropriation of the cultural tools that mediate mathematical activity (Brown, Collins, & Duguid, 1989; Forman, 2003). The goal of this article is to illustrate one aspect of tool appropriation that is brought to the foreground by an analytic focus on the role of embodiment in tool use. Through two microethnographic case studies, we first outline certain features of the emergent development of perceptuomotor integration with a mathematical museum exhibit, and then we explicate how varying degrees of fluency with an instrument provide one with the experiential resources to imagine the enactment of bodily orientations with the instrument to do novel mathematical tasks. By describing emergent mathematical tool fluency in terms of perceptuomotor integration, we purposefully eschew accounts of mathematics learning that appeal to hypothetical mental structures or categories of reasoning that are divorced from the irreducible richness of mathematical experience.

Perceptuomotor Integration in Informal Mathematics Learning

In resonance with research in experimental and developmental psychology, cognitive science, and neuroscience, we suggest that the development of tool fluency entails the interpenetration of the perceptual and motor aspects of an activity, allowing the performer to act with a holistic sense of unity and flow. It has been demonstrated, for example, that when professional pianists listen to a piano piece they involuntarily enact the finger arrangements corresponding to the notes they hear (Haueisen & Knösche, 2001). Conversely, a relative lack of fluency is often sensed by failing to achieve such holistic integration. For example, as one learns a second language one commonly experiences a separation between the motor articulation of an unfamiliar sound and the perceptual qualities of the sound that gets produced. Similarly, in learning to play a string musical instrument there is a common mismatch between the sound that one is supposed to generate (i.e., a sound that one hears others producing) and one's fingering on the strings. Similar examples can be found easily in sports, crafts, drama, and other fields. We refer to the achievement of such intertwining of perceptual and motor aspects of tool use as *perceptuomotor integration*.

Through two microanalytic case studies of visitors to a San Diego science museum, this study aims to illustrate the ways in which perceptuomotor integration emerges and partially constitutes mathematical learning. The case studies focus on the use of a particular *mathematical instrument*. A mathematical instrument, we propose, is a material and semiotic tool together with a set of embodied practices for its use within the discipline of mathematics. Fluent use of a mathematical instrument allows for culturally recognizable creation in mathematical domains, just as musical instruments enable practitioners to produce distinct kinds of music that members of musical communities acknowledge. Drawing in Motion, the mathematical instrument that is the focus of this study, is a prototype exhibit that requires physical engagement and collaboration between two people who jointly produce a graph on a displayed Cartesian coordinate plane. We found that emerging fluency with this exhibit enabled visitors to play with the mathematics of certain parametric functions depicted graphically. In the following analyses we trace the emergence of perceptuomotor integration during the use of the exhibit and the role of this emerging fluency in visitors' imaginative enactments during interviews.

Background

Research on the Understanding of Functions

The topic of functions has long been considered a key element of the K–12 mathematics curriculum (National Council of Teachers of Mathematics, 2000). Researchers have examined learning about functions from multiple points of view, such as: (a) identifying different ways of thinking about functions (Carlson, 1998; Confrey & Smith, 1995; Schoenfeld, Smith, & Arcavi, 1993; Sfard, 1991; Thompson, 1994b; Yerushalmy, 1997); (b) documenting errors and misconceptions about functions (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Goldenberg, 1988; Herscovics, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Tall & Vinner, 1981); and (c) examining the use of technology for the learning and teaching of functions (for an overview, see Heid & Blume, 2008b, pp. 79–84). Most of the studied technologies are computer-based (e.g., Goldenberg, 1988; Magidson, 2005; Moschkovich, 1996, 1998; Schoenfeld et al., 1993; Yerushalmy & Schwartz, 1993), but others are mechanical devices (e.g., Hines, 2002; Izsák, 2000; Meira, 1998; Piaget, Grize, Szeminska, & Bang, 1977/1968).

Regarding different ways of thinking about functions, many researchers have articulated from different theoretical frameworks, a contrast between dynamic/processual and static/structural conceptions. Sfard (1991) proposed an epistemological duality of process/object, characterizing the latter as a static/structural reification of the former. Based on case studies, Monk (1992) described the dynamic/static opposition by distinguishing between *Across-Time* and *Pointwise* conceptions of functions. According to Monk (1992) students with a Pointwise conception are unable to imagine the function dynamically. Researchers focused on rate of change as the main component of the Across-Time conceptions (Thompson, 1994a). Moschkovich, Schoenfeld, and Arcavi (1993) introduced the notion of *Cartesian Connection* to account for the ability to envision solutions to a linear equation as ordered pairs corresponding to points in a Cartesian graph, which is an essential aspect of Pointwise conceptions. The distinction between covariation and correspondence is also related to the dynamic/static opposition. Carlson, Jacobs, Coe, Larsen, and Hsu (2002) defined "*covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Covariational reasoning is viewed as essential to mastering the concepts of rate of change and slope (Confrey & Smith, 1994; Lehrer, Strom, & Confrey, 2002; Lobato & Ellis, 2010; Lobato, Ellis, & Muñoz, 2003; Lobato, Rhodehamel, & Hohensee, 2012; Lobato & Thanheiser, 2002; Saldanha & Thompson, 1998; Simon & Blume, 1994; Stump, 2002; Thompson, 1994b). Confrey and Smith (1994) described correspondence as the more conventional and static approach to functions: "conventional treatments of functions start by building a rule of correspondence between *x*-values and *y*-values, typically by creating an equation of the form $y = f(x)$" (p. 135).

Given this brief overview of the literature on the understanding of functions, the present paper is distinctive in at least two regards: (a) an analytical approach centered on the phenomenology of lived experience; and (b) a mathematical focus on parametric functions. We subsequently elaborate on each of these.

Phenomenology of lived experience. Most of the literature that we reviewed examines students' talk and actions with the goal of identifying their mental schemes, forms of reasoning, cognitive structures, procedural strategies, and the like. Typically, a record of student activity comes to be described as the enactment of a particular form of reasoning, exhibiting the mental possession of, say, covariational reasoning or the Cartesian Connection. We think that this kind of description for students' actions and talk, while having the appearance of being neutral and simple, tends to pass on assumptions of mentalism and it may distort the students' experiences in significant ways. Many of these potential distortions have been described by theorists of situated cognition (Brown et al., 1989; Greeno & the Middle School Mathematics Through Applications Project Group, 1998; Lave & Wenger, 1991). Instead, we work to describe *lived experiences* in terms of temporal flows of perceptuomotor activity, which are at once bodily, emotional, and interpersonal. Our view contributes to an emerging body of work in research in mathematics education aiming to understand how multiple streams of embodied activity constitute experiences of mathematical thinking and learning (Arzarello, 2006; Radford, 2009; Roth, 2011). From our perspective, the temporal flow of perceptuomotor activity cannot be characterized in terms of self-contained structures, schemes, or patterns, and necessitates an irreducible immersion in the particulars of the actors' gestures, tone of voice, gaze, facial expression, steadily emerging from their creative being-in-the-world. The goal of our microanalysis is to grasp the ever-changing way of being in the world lived by the participants at a particular place (Nemirovsky, 2005), as they incorporate the actual or imagined presence of tools and others.

Parametric functions. Parametric functions are families of functions that have one or more parameters in common. Sometimes they are referred to as parametric "equations" because the standard way of defining these functions is through particular equations.

> Parametric equations are a set of equations that express a set of quantities as explicit functions of a number of independent variables, known as "parameters." For example, while the equation of a circle in Cartesian coordinates can be given by $r^2 = x^2 + y^2$, one set of parametric equations for the circle are given by: $x = r \cos(t)$ [and] $y = r \sin(t)$ . . . . Note that parametric representations are generally nonunique, so the same quantities may be expressed by a number of different parameterizations. A single parameter is usually represented with the parameter t, while the symbols u and v are commonly used for parametric equations in two parameters. (Weisstein, n.d)

Individual parametric functions can also be dependent on other variables, not called *parameters*, that may not be common to all of them. By composing parametric functions, one may obtain a relation rather than a function, such that the graphical expression of a pair of parametric functions can be a multivalued curve, so that a single value on the abscissa may correspond to many ordinate values. In our study the parameter is time and each function describes the motion of a different person (for a study using an analogous device see Noble, DiMattia, Nemirovsky, & Barros, 2006); in other words, graphs are obtained by pairing two functions of position versus time, one for the horizontal axis and another for the vertical axis. Each parametric function elicits its own *motion-based narrative* (Nemirovsky, 1996), and their composition expresses the interactive coordination between collaborating users.

The passage from working with individual functions of time to composing a family of functions of time is one we will refer to as *parameterizing time*. On the basis of the two distinctive features we described previously—phenomenology of lived experience and parametric functions—we suggest that parameterizing time can be important for teaching and learning about temporal functions. Although we experience time always flowing in all moments of life, lived time appears elusive and intangible. As soon as we try to point at lived time, we sense that it is gone; we can only point at it indirectly, by means of events that took place "then," including displays of a clock face or locations on a time line. That is not the case for many other quantities, for instance, position: We easily can point at a position where something is or would be. The elusiveness of lived time poses great complexities in learning to use temporal functions (Yerushalmy & Shternberg, 2001), some of which, we suggest, may be grappled with by parameterizing time. In the cases reported here, rather than where someone is at a certain time, the issue is the position of someone *while* another person is somewhere else.

Mathematical Instruments: An Embodied Perspective on Mathematical Tool Use

Embodied cognition and tool use. We define a mathematical instrument as a material and semiotic device together with a set of embodied practices that enable the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) that are acknowledged within the culture of mathematics. The role of technology, tools, and artifacts in mathematical thinking and learning have been, and continue to be, a major area of educational research (Blume & Heid, 2008; Heid & Blume, 2008a; Hoyles & Lagrange, 2010). Researchers have taken a variety of theoretical approaches to this issue, including, most notably, various strands of Piagetian constructivism (Schliemann, 2002), notions of semiotic mediation from sociocultural theory (Chassapis, 1999), and instrumental approaches (Drijvers et al., 2010).

Our goal in this study is to articulate an emerging, novel perspective on the age-old use of tools in mathematics. In particular, we advocate for a perspective on tool fluency that is explicitly informed by an embodied approach to mathematical thinking and learning, a perspective that, for us, entails that (a) mathematical thinking is constituted by bodily activity at varying degrees of overt and covert expression, and (b) mathematical learning consists of transformations in learners' lived bodily engagement in mathematical practices (Nemirovsky & Ferrara, 2009; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Varela, Thompson, & Rosch, 1991). (1961)

To be clear, although many scholars, such as Piaget (1961) and his intellectual descendants, acknowledge sensorimotor activity as a necessary *precursor* to genuinely mathematical thought, the brand of embodied cognition that inspires our work aims to question dualisms between body and mind, between "outer" and "inner," between perceptual and conceptual, and, most importantly for our purposes, between bodily, tool-mediated expression and mental structures or schemes that many mathematics education scholars see as the site of "mathematical understanding." It is for this reason that we choose to use the term *instrument*, a word that intentionally connotes the culture of music; just as one does not speak of a violinist's expertise as something divorced from the quick movements of her fingers over the strings and the trained dance of her eyes across a musical score, we suggest that it is equally objectionable to cleave mathematical expertise from the skillful motoric and perceptual engagement with the tools of the discipline.

Comparison with other theoretical approaches to mathematical tool use. To better illustrate the approach taken in this study, it is helpful to compare our definition of an instrument with the theory of instrumental genesis articulated by Verillon and Rabardel (1995) and others (Artigue, 2002; Drijvers & Gravemeijer, 2005; Guin & Trouche, 1999; Trouche, 2005). These scholars differentiate between the physically instantiated artifact and the instrument, a "mixed entity" (Artigue, 2002, p. 250) comprising the physical device along with mental schemes that detail its use in particular situation types. Through the process of *instrumental genesis*, the artifact becomes an instrument as the user (a) transforms the artifact to fit her purposes and (b) develops a suite of mental schemes specifying the artifact's use under certain conditions (Verillon & Rabardel, 1995). Using this approach, various researchers have studied the use of symbolic calculators and computer algebra systems in the classroom, documenting the mental schemes associated with instrumental genesis (Drijvers & Gravemeijer, 2005; Guin & Trouche, 1999; Trouche, 2005).

The theory of instrumental genesis is, thus, a *dialectical* approach, one that aims to navigate correspondences "between action and conceptualization" (Trouche, 2005, p. 155). Indeed, a dialectical approach to tool theory is not unique to the theory of instrumental genesis:

Vygotskian mediational theory essentially relies on a similar epistemological geography in the form of intrapersonal and interpersonal planes that are navigated theoretically through a process of internalization (Vygotsky, 1978, 1987). For example, some researchers draw on Vygotsky's notions of tool mediation and internalization to investigate the role of mathematical instruments—including compasses, abaci, and Dürer's glasses—in the development of mathematical expertise in primary school classrooms (Bartolini Bussi & Boni, 2003; Chassapis, 1999; Maschietto & Bartolini Bussi, 2009). Although the present study resonates with work by Bartolini Bussi and Boni (2003), Chassapis (1999), and Maschietto and Bartolini Bussi (2009) in its emphasis on multimodality, these researchers characterize the development of tool fluency and mathematical expertise in terms of a process of internalization by which concrete-material artifacts are transformed into abstract-mental representations, a characterization that we try to deconstruct.

Nondualist approach to tool use. Although we agree with proponents of the instrumental genesis theory who wish to overcome the "technical–conceptual cut" (Artigue, 2002, p. 247) our commitment to embodiment inspires us to achieve this theoretical desideratum through a nondualist approach rather than a dialectical one. In other words, we aim to depart from theories that delineate between inferred mental representations, such as schemes, and bodily engagement with physical and semiotic artifacts; all too often we feel these approaches have a tendency to privilege mental structures by allowing the category of mathematical understanding to be exhausted by them. Instead, we locate tool fluency as well as mathematical thinking and learning in the process of perceptuomotor integration when learners engage with others and physical artifacts. We propose that mathematical knowing is constituted by—not dialectically related to—embodied tool use. This means that our analytic project is not one of charting correspondences between visible bodily activity and inferred, formal mathematical thought processes; instead, we take mathematics learning to inhere in transformations in lived bodily experience.

Some researchers achieve a nondualist approach to mathematical instruments by emphasizing the lived experiences of emerging tool fluency and use. For example, Noble, DiMattia, Nemirovsky, and Barros (2006) suggest that gaining competency with a mathematical instrument is akin to changing one's *lived-in space*. Based on a sequence of microanalytic case studies of high school students engaging with two mathematical instruments—including a kind of analogue version of Drawing in Motion—the authors highlight three aspects that help to account for the complexity of learning to use a new tool or instrument: Tools are constituted by use, bodies incorporate tools, and developing tool fluency constitutes a change in the user's lived-in spaces.

> In other words, the world changes for the student as she or he learns to use a given tool. We argue that one's lived-in space is transformed through tool use and that barriers between one's body and the tool become less relevant. One's own sensitivities and perceptions and one's own physical abilities are shaped through one's growing competence with a tool. (p. 434)

Our work extends this analysis by foregrounding and explicating perceptuomotor integration as a hallmark of one's experience of tool fluency.

Our attempt to overcome mind-body dualism is not about ignoring one or the other, but about reformulating the themes we want to examine—learning, teaching, mathematical practices, use of instruments, and so forth—from an alternative perspective according to which the split between mind and body does not arise. This perspective focuses on lived experience, understood as a temporal flow of perceptuomotor activity. The phrase “temporal flow” is essential because it implies that any perceptuomotor activity is infused with past and future, so that actions pursued at a certain moment cannot be isolated to whatever is physically “there” at that moment. Perceptuomotor activity is always permeated by expectations, recollections, fantasies, moods, and so on. Imagine someone obtaining the roots of a polynomial on paper. Rather than describing her reading and writing acts, including the movements of her eyes, hands, and vocal chords as outer manifestations of her mental schemes for the concepts of polynomial and root, we see these acts as partial enactments of extended and shifting courses of action that she experiences, saturated with feelings and puzzlements. If she wrote, for example, a general expression for a polynomial of degree n, we do not think of her perceiving this combination of marks and of her projecting a mental scheme on them. Rather, we strive to understand the temporally extended courses of action that she experiences in the acts of writing and perceiving the expression.

Other contemporary research studies address the kind of phenomenon we aim to capture with the notion of a mathematical instrument, although they do not always use the analytic framework we propose in this study. For example, we suggest that dynamic geometry environments (DGE’s) constitute a family of mathematical instruments. The ability to “drag” components dynamically in a geometric sketch has piqued interest in exploring the ways in which dragging might mediate between a DGE’s perceptual features and the formal rigor of traditional geometric proof (Arzarello, Olivero, Paola, & Robutti, 2002; Leung, 2008; Lopez-Real & Leung, 2006). Dragging in a DGE can support, from our perspective, an important type of integration of motoric (i.e., dragging a geometric element by hand) and perceptual (i.e., visual consequences on the whole diagram) aspects. Another family of mathematical instruments that has received attention from researchers is motion detectors that permit students to explore the modeling of their own body movement through space by means of real-time graphical displays (Arzarello, Pezzi, & Robutti, 2007; Nemirovsky, Tierney, & Wright, 1998; Robutti, 2006). In this study we explore another category of mathematical instrument: museum exhibits that engage learners in mathematics through kinesthetic activity.

Informal Mathematics Learning

In recent decades, the educational role of museums and other informal settings has become increasingly recognized by educators, researchers, and policy makers (Falk & Dierking, 1992; Hein, 1998; National Research Council, 2009). Although there is currently a well-established body of literature related to informal science learning, this article contributes to a small, but growing, field of research on the nature of learning mathematics in informal settings (Cooper, 2011; Guberman, Flexer, Flexer, & Topping, 1999; Gyllenhaal, 2006; Mokros, 2006). Although for many years the classic exhibition *Mathematica*, developed by Charles and Ray Eames and first installed in 1961 in Los Angeles, was the lone example of a major mathematics exhibition, science museums are rapidly developing mathematically oriented exhibits and experiences. Researchers have argued that the inclusion of mathematical activities in museums, science centers, and other informal environments has the potential to complement formal learning in school mathematics classrooms, address educational inequities, promote positive attitudes towards mathematics, and bolster recall of specific mathematical content learned in school (Gyllenhaal, 2006; Mokros, 2006). This study aims to contribute to this growing understanding by providing a detailed, empirically based exploration of the microgenesis of mathematical expertise in a science center. Moreover, in situating itself within an embodied cognition approach, this study contributes to another growing area of research that explores the ways in which visitors constitute their own experiences through embodied social interaction and engagement with objects (Heath & vom Lehn, 2004; Rahm, 2004; vom Lehn, Heath, & Hindmarsh, 2001).

Perceptuomotor Integration

It is a common experience that in learning a new skill one goes through phases in which perceptual and motor aspects of the activity seem to be discordant. The adult learner of a second language, for instance, often feels incongruence of her motor arrangements of vocal chords, lips, and tongue, with the sounds that she hears herself producing. The same can be said of learning basketball, salsa dancing, guitar playing, car driving, and countless other activities. We suggest that the transition from discordance between perceptual and motor aspects to their integration is common to all learning; that is, perceptuomotor integration is a milestone for fluency in any field.

We begin by illustrating two main qualities of fluency as perceptuomotor integration. First, motor activity is involuntarily enacted as part of perceiving. Second, partial motor and perceptual components have the power to elicit the enactment of the activity as a whole over time. For example, consider the following description of musical imagery:

> With music I know well, such as Chopin's mazurkas, which I learned by heart 60 years ago and have continued to love ever since, I have only to glance at a score or think of a particular mazurka (an opus number will set me off) and the mazurka will start to play in my mind. I not only "hear" the music, but I "see" my hands on the keyboard before me, and "feel" them playing the piece—a virtual performance which, once started, seems to unfold or proceed by itself. (Sacks, 2008, p. 34)

That motor activity is involuntarily enacted as part of perception is stressed by Sacks' observation, "I not only 'hear' the music, but I 'see' my hands on the keyboard before me, and 'feel' them playing the piece." In addition, that partial motor and perceptual components have the power to elicit the enactment of the activity as a whole is reflected in the comment, "I have only to glance at a score or think of a particular mazurka (an opus number will set me off) and the mazurka will start to play in my mind." These phenomena are not only familiar to any fluent player; they are also documented in the literature on neuroscience. For example, Haueisen and Knösche (2001), comparing magnetoencephalography recordings of expert pianists and nonpianists listening to a piece of piano music, found significant involuntary activation in the contralateral primary motor cortexes of expert pianists but not in novices.

Observations concerning these two qualities of perceptuomotor integration have been documented in many other fields, such as basketball playing (Aglioti, Cesari, Romani, & Urgesi, 2008), dancing (Calvo-Merino, Glaser, Grèzes, Passingham, & Haggard, 2005; Calvo-Merino, Grèzes, Glaser, Passingham, & Haggard, 2006), and drawing (Beets, Rösler, & Fiehler, 2010). One of the most striking examples of perceptuomotor integration is the experiments with subjects who wore lenses shifting or inverting their visual field and who went through a process of "normalizing" vision on the basis of regular motor activity (Redding, Rossetti, & Wallace, 2005; Sekiyama, Miyauchi, Imaruoka, Egusa, & Tashiro, 2000). After some time conducting regular activities wearing these lenses, which can be very difficult at first, the surroundings start to be seen as "normal;" furthermore, taking off the lenses after adaptation demands a new period of training to recover regular visuo-motor activity. In some cases, the achievement of perceptuomotor integration demands multiyear intensive training (e.g., learning a second language in adult age), but, for other more limited tasks, it can be achieved in relatively short time periods (e.g., mirror drawing, that is, drawing while looking at one's hand through a mirror). There are multiple indications that the involuntary motor participation in perception is crucial, in particular, to one's ability to predict and anticipate events (Aglioti et al., 2008; Candidi, Leone-Fernandez, Barber, Carreiras, & Aglioti, 2010; Llinás, 2002).

A third aspect of perceptuomotor integration that deserves mention is its relationship with sociocultural factors. In our analysis, we suggest that, while perceptuomotor integration constitutes a transformation that is experienced by an individual, it is (a) shaped by relatively local social interaction and relatively global cultural factors and (b) socially consequential because one's degree of instrumental fluency has bearing on one's membership to various social groups. For example, Sacks' ability to play fluently a mazurka on the piano (a) is structured, presumably, by a history of interactions with his piano instructors as well as cultural norms for musical notation and piano performance and (b) makes him more likely to be invited to perform at, say, a wedding than would a piano novice. In fact, a close analysis of the process of perceptuomotor integration in our study suggests that an a priori distinction between individual and interpersonal processes does not characterize mathematical learning experiences.

The Study

This study was part of a larger project called Math Core for Museums, a collaboration among exhibit developers and researchers at the Science Museum of Minnesota in Saint Paul, MN; the Museum of Science in Boston, MA; Explora in Albuquerque, NM; the North Carolina Museum of Life and Science in Durham NC; TERC in Cambridge, MA; and the Center for Research in Mathematics and Science Education in San Diego, CA. Goals of this project include the development and evaluation of interactive exhibits focused on the mathematics of ratio and proportion as well as ongoing research on the nature of informal mathematics learning. In particular, the research component of this project aims to investigate the various ways in which visitors' verbal and bodily activities reveal and constitute their emerging understandings about ratio and proportion.

During the early prototyping phases of Math Core for Museums, the Oregon Museum of Science and Industry in Portland, OR coordinated with the Center for Research in Mathematics and Science Education to install an exhibit prototype, Drawing in Motion, at the Reuben H. Fleet Science Center in San Diego, CA. The data for the current study resulted from video recordings of visitor interactions with Drawing in Motion and subsequent interviews with them. Drawing in Motion, developed and manufactured by the Oregon Museum of Science and Industry, is an interactive exhibit that requires the collaboration of two visitors. Each visitor controls the motion of a handle along a 3-foot linear scale, corresponding to a graphical vertical or horizontal axis (see Figure 1). A large LCD screen displays a cursor controlled by the two handles that determine the *x*- and *y*-coordinates of the cursor. The two participants jointly draw on the screen by moving the handles across two wooden panels. Both the digital screen and the wooden panels where the handles are located display numerical coordinates that correspond to one another.

The exhibit includes three challenges as well as a free-drawing mode. Challenge 1 requires the users to trace a rectangle, reaching each of the following vertices in order: (1, 1), (1, 6), (9, 6), (9, 1). Challenge 2 requires that users trace a line segment with slope 1, beginning at the point (1, 1) and ending at the point (7, 7). Challenge 3 requires users to trace a right triangle with a hypotenuse that has slope 2, reaching each of the following vertices in order: (9, 9), (9, 1), (5, 1), (9, 9). Each challenge includes several visual and verbal aids, such as indications of the particular point that needs to be attained and language that suggests how the visitors should move in relation to one another. Each point is indicated by a coordinate pair and a dot that appears at the points in the order in which they must be attained.

Method

Data Collection

Drawing in Motion was installed in a secluded area of the Reuben H. Fleet Science Center, and only research staff and study participants had access to the exhibit area. The exhibit was installed as part of the ongoing design-based research that helps guide exhibit development in museum settings. It is standard practice for exhibit developers and researchers to study visitor engagement with unfinished prototypes in order to inform future design decisions as well as to study learning in informal settings. The researchers did not design the exhibit, but selected it for installation because it (a) was designed to engage visitors in the mathematics of functions, (b) elicited bodily interaction, and (c) met science-center design standards of being both open ended and accessible to a broad audience.

Members of the research team personally recruited participants by approaching groups that had already come to the science center and asking whether they would like to participate in a research study. Thus, all participant groups were friends or family members that had come to the science center together. We targeted groups with children aged 7–14 because Drawing in Motion was designed for children in this age range and their families. We also included both younger participants and all-adult groups in the event that children in this age range were not present in the museum. Each participating group engaged with the exhibit and then participated in an informal interview.

Data were collected from 17 different visitor groups, including both adult participants and children as young as 5. Although some visitor groups consisted only of adults, other groups consisted of a child-parent dyad or sibling pairs accompanied by parents. Each visitor group was invited to engage with Drawing in Motion for as long as they liked, with exhibit experiences ranging from 4 to 23 minutes. It is worth noting that, although this may seem like a short amount of time from the perspective of school-based educational research, this range represents an unusually long amount of time from the perspective of museum studies and informal education (Falk & Dierking, 1992). Groups with more than two members would sometimes take turns using the exhibit. Some parents accompanying sibling pairs assisted the children, whereas others remained more at the periphery of the activity. The visitors were offered minimal introduction to the exhibit, although members of the research team occasionally offered assistance when it was clearly needed or explicitly requested. While engaging with the exhibit, visitors were video recorded from two camera angles in order to provide adequate data for the analysis of gesture and other physical behaviors. In addition, we collected a video screen capture of everything displayed on the LCD-monitor of the exhibit throughout the visitors' entire engagement with the exhibit.

Following their engagement with Drawing in Motion, visitor groups collectively participated in an informal stimulated-recall interview with one of the researchers following their interaction with the exhibit. Interviews were video recorded and ranged from 8 to 23 minutes. Although some parents chose to participate in sibling interviews, others chose not to. The video screen capture was made available throughout the subsequent interview, and much of the interview consisted of an informal, improvisatory conversation. Because of this, no two interviews included identical sets of questions. However, a general format for the interviews can be described. In a typical interview, a member of the research team first asked the visitors to share their general impressions of the exhibit, whether they enjoyed the experience, and whether the exhibit reminded them of anything. The interviewer then played back the screen capture taken during the exhibit experience, pausing the playback intermittently to ask participants to offer explanations for the appearances of their various graphical productions and to reenact the necessary handle movements to reproduce or improve them. For example, if playback of the visitors' attempt at Challenge 2 showed a deviation from the target line, the interviewer might ask, "What happened with your handles to cause the cursor to move away from the line like that?" or "What would you do next time to draw the line more accurately?"

Over the course of the screen-capture playback, participants frequently also were asked to enact or describe the handle movements required to produce novel graphical images. For example, during several interviews, the interviewer asked how the visitors would need to move the handles of Drawing in Motion to produce relatively steeper or shallower lines. Other novel tasks were posed in response to the emerging conversation. For example, the second episode described in this article was preceded by a conversation in which the participants explained to the interviewer that images requiring only vertical or horizontal lines are the easiest to draw on the device. The interviewer responded by posing the question of how to draw a diagonal—and hence, more challenging—line.

At the end of the interview, participants were again invited to share their overall impressions of the exhibit, to discuss which aspects of the experience were challenging, to offer design suggestions, and to identify prior experiences—both mathematical and nonmathematical—that they believed were related to their exhibit experience in some way. For example, the interviewer might ask, "What would you change about this exhibit?"

Episode Selection

Following data collection, all video data (approximately 7.5 hours of observation and interview) were reviewed and two microanalytic case studies were selected for analysis. As with any case study approach, multiple selection criteria may be appropriate, depending on the nature of the research questions. Because we were interested in emergent instrumental expertise as an embodied phenomenon, we chose our episodes according to several criteria that were appropriate for this goal. First, we sought to select episodes that represented a range of mathematical complexity and illustrated evolving perceptuomotor expertise. Thus, the first episode consists of a sibling pair just beginning to gain fluency with the exhibit as they trace the rectangle indicated in Challenge 1. In the second episode, a different pair of siblings demonstrates emergent expertise as they explain to an interviewer how they would approach a novel problem requiring the creation of a negatively sloped diagonal line. Together, the two episodes trace a plausible trajectory of mathematical learning through the development of tool fluency. The first episode illustrates early stages of perceptuomotor integration with a relatively simple mathematical object, whereas the second episode demonstrates a more established level of expertise with Drawing in Motion in relation to a mathematical object that is more challenging to produce with the device.

Of course, many episodes might have fit this criterion, but our final selections were also based on a second criterion that privileged multimodal *richness.* Episodes that included a relatively greater variety of behavioral data—including gesture, talk, body movement, prosody, and exhibit manipulations—were favored, because these provide the microanalyst significantly more information about visitors' ongoing experiences. In selecting behaviorally rich episodes, our goal was not representativeness but rather the enrichment of the reader's own perception of learning phenomena through immersion in the details of a complex moment.

Analytical Framework: Microethnography and Husserl's Philosophy of Experiential Time

Following Streeck and Mehus (2005) we have chosen to characterize the methodology of this study as *microethnographic*: "the microscopic analysis of naturally occurring human activities and interactions" (p. 381). We understand microethnography to be a type of interaction analysis (Erickson, 1996, 2004; Goodwin, 2003; Hall & Stevens, 1995; Stivers & Sidnell, 2005), distinguished by the study of multimodal strands of activity (e.g., talk, facial expression, gesture, gaze) over short periods of time, and by its attention to the cultural and lived circumstances of the studied events, striving to achieve "thick descriptions" (Geertz, 1973). Talk, gesture, facial expression, body posture, drawing of symbols, manipulation of tools, pointing, pace, and gaze are examples of modalities to be traced (Erickson, 1996, 2004; Goodwin, 2003; Hall & Stevens, 1995; Stivers & Sidnell, 2005; Streeck & Mehus, 2005). The growing interest in the study of gestures in mathematics learning and conceptualization (Alibali & DiRusso, 1999; Nemirovsky & Ferrara, 2009; Radford, 2003, 2009) has increased the recognition of microethnography as a suitable research methodology. Furthermore, a small but growing body of research uses microethnography to trace the detailed ways in which visitors navigate physical and social interactions in museums (Heath & vom Lehn, 2004; Rahm, 2004; vom Lehn et al., 2001).

To further clarify the kind of microanalytic approach taken in this study, we find it useful to appeal to Husserl's (1991) phenomenological framework for experiential time. We use this framework to guide our investigation of participants' experiences over a particular stretch of time. From this perspective, all temporally extended experiences—for example, the experience of hearing a particular melody—are constituted by an ever-changing flow of *retentions*, *now phases*, and *protentions*. For instance, Husserl argues that the experience of a particular note in a melody is not merely determined by the presently sounding tone; instead, the act of perceiving the note also includes the retention of notes sounding just prior and the protention, or immediate anticipation, of notes that are about to sound. The same note may sound completely different depending on its "just past" and "about to happen." More generally, the experiential present is a type of ideal limit, ephemerally constituted by a lived stream of retentions and protentions.

We can think of the note in a melody as akin to a particular act or utterance during a conversation. Just as the experience of a note is partly constituted by the sounds that precede and follow it, a participant's action in an interaction is to be understood with reference to what she sensed in her "just past" and in her "about to happen." For example, a participant who, while speaking, puts a hand up in the form of a wait gesture, may be (a) responding to the sense that her coparticipants seem eager to say something and (b) anticipating that her coparticipants might respond to her gesture by delaying their talk a little longer, until she will have completed her utterance.

As microethnographers with the goal of producing rich phenomenological descriptions of visitor experiences, our analytic method can be characterized as the development of a detailed account of this ongoing flux of retentions and protentions as they unfold for a participant over the course of a short period of time. Thus, for each of the selected episodes, we scrutinized gestures, utterances, or eye-gaze shifts as evidence of a significant change in the participant's experiential present. At each of these significant moments, we used the rich store of available behavioral data to develop possible descriptions of the retentions and protentions constituting that particular experiential present. Although these descriptions are based on repeat scrutiny of an empirical video record, they are, of course, interpretive and provisional. In presenting our results we aim to offer the reader both our analyses and sufficient descriptive granularity so that the reader can draw her own interpretation of the data.

Microanalysis Results

In this section we present our analysis of two episodes that we name by their participants: (a) Ivan and Kayla, and (b) Rafaela and Nicolas. These are two different pairs of siblings. Whereas the Ivan and Kayla episode encompasses their initial encounter with Drawing in Motion*,* the Rafaela and Nicolas episode is an excerpt from their interview after they had finished using the exhibit. The mother of Ivan and Keila observed their activities but did not participate. The parents and a little sister of Rafaela and Nicolas accompanied them. During the interview their little sister stood next to them from time to time, and their parents went to use *Drawing in Motion* on their own. For each episode we include the corresponding transcription twice. The initial version of the transcript is included as a whole—verbatim, and with little detail—just to give the reader a sense of the talk throughout the episode. The subsequent version of the same transcript is much more detailed and interspersed with graphic clarifications and textual interpretations.

Real-Time Emergence of Perceptuomotor Integration: Ivan and Kayla

In this 42-second episode, Ivan (age 12) and his sister, Kayla (age 10), began their exhibit experience with a first attempt at Challenge 1. A member of the research team had just offered them minimal instruction on the exhibit's use, suggesting that they engage with the exhibit in whatever way made sense to them and demonstrating to them the use of the "clear" button. Without any explicit planning, Ivan used the *y*-axis handle and Kayla the *x*-axis handle. During this episode Ivan and Kayla verbally communicated very little with one another, encouraging us to attend carefully to subtle shifts in body motion and eye gaze.

A verbal transcription of the episode follows. Numbers in parentheses indicate pause duration in seconds. Bolded speech indicates overlapping speech, that is, speech produced by the participants at the same time. Italicized remarks enclosed in square brackets include descriptive commentary of prosodic features and nonverbal participation.

(2.1)

I: move **here one** point

K: **okay**

(4.7)

I: okay now um one point six

(14.9)

I: Kayla, you're **going the** wrong way

K: **what**

(0.6)

K: no (0.1) way [*hard exhale, as though laughing*]

I: 'kay go

(2.0)

I: nope the other way the other way the other way the other way

(2.0)

I: kay (2.0) now g' the other way

(3.2)

We view this episode as an illustration of the relatively early stages of perceptuomotor learning. To outline a fine-grained theoretical framework for this process, we appeal to Husserl's philosophy of experiential time. We suggest that, in early perceptuomotor learning phases, the just-past and the about-to are relatively cleanly associable with perceptual and motor aspects respectively (or vice versa). On the other hand, as perceptuomotor fluency emerges, greater interpenetration of perceptual and motor aspects is revealed by retentions and protentions that each includes both perceptual and motor aspects.

Annotated episode. Below is a detailed transcript, accompanied by annotations of those aspects of the episode that are particularly illustrative of our emerging theory of perceptuomotor learning. Underlined words indicate the speech or physical action corresponding to the provided video frame. In the video images, dotted arrows indicate eye gaze, solid arrows indicate body motion and handle movements, and circles indicate the portion of the graph produced during the corresponding time segment.

(2.1)

[*Ivan presses the clear button and clasps the y-axis handle near the number 7. Kayla looks at Ivan as he presses the clear button, clasping her handle with both hands, near the number 5. She briefly glances at the monitor, then turns her gaze toward the panel buttons, resting her left hand on the panel.*]

1. I: move (Figure 2) **here** [*Kayla turns her gaze to the monitor. Ivan turns his gaze to the numbers on the panel, looking toward the higher numbers on the left side of the panel.*]

Ivan's utterance "move here" in Line 2 is echoing the words "move here" that were displayed on the screen just a moment ago, along with an image of a hand pointing to the point (1, 1). Drawing on a wealth of background expectations, he senses that the location on the graph is connected to information on the panel; he has read the words on the display and is now saying them aloud while looking toward the panel. That he is looking for the *1* on the left side of the panel may also be partly due to past experiences with number lines in which the lower numbers would be placed on the left side. Ivan's just-past includes the prompt to "move here." By saying the words aloud, he evokes the memory of the prompt so that the scope of his experiential present includes it. Looking toward the panel, Ivan's about-to includes looking for the "here" of "move here" on the *y*-axis panel.

2. *I*: **one** point [*Ivan sweeps his gaze along the panel to the right. As he says, "point," he shifts his body weight to the right.*]

   K: **okay** [*Kayla lifts her hand from the panel and moves it to the right. After she says, "okay," she begins to move her handle a little to the left, creating a small horizontal line segment.*]

Ivan's just-past includes having looked unsuccessfully for the "1" on the left side of the panel. Seeing the "1" on the right side of the panel now, his about-to includes getting ready to "move here."

3. (0.2) [*Ivan begins to move the handle to the right. After about 200 ms of continuing to look at the panel, he shifts his gaze (Figure 3) to the monitor.*]

Ivan's just-past includes having found the *1* and preparing to achieve his goal of attaining the *1*. His present includes moving the handle in the direction of the found *1*. Shifting his gaze to the screen, he is about to observe an expected relationship between his handle movement and events on the screen. These exchanges illustrate how in the early stages of perceptuomotor learning, the just-past and the about-to are more cleanly associable with perceptual and motor aspects: At this moment, Ivan's just-past is primarily motor, consisting of his manipulation of the handle, and his about-to is predominantly perceptual, consisting of visual apprehension of events on the computer monitor.

4. (2.2) [*Ivan continues to move his handle to the right and steps to the right. Kayla moves her handle to the left, pausing intermittently. They both look at the screen (Figure 4).*]

5. (1.0) [*Ivan moves the handle below a* y*-coordinate of 1 so that the cursor has a* y*-coordinate of about 0.5. Ivan pauses the handle, shifts his weight to the right, <u>and looks down at the right end of the panel</u> (Figure 5). Kayla pauses, then moves her handle to the left, producing a horizontal segment. She continues to watch the screen.*]

One possibility is that, aware of the goal to arrive to the point (1, 1), Ivan continues to move his handle past the coordinate "1" with the intention of reducing the horizontal distance between the cursor and the target. Because his motor action during Line 5 was cotimed with a graphical trajectory that includes both vertical and horizontal displacement, it is reasonable for Ivan to expect that he has control over the horizontal displacement. In Line 6, Ivan has overshot his goal and moving his handle further to the right does not appear to get him any closer to "here." He may now be looking down to ascertain where the handle is relative to the numbers on the panel, because he has also established that "here" is a location on the panel.

6. (1.3) [*Both Ivan and Kayla are now making small movements with the handles. <u>Ivan shifts his gaze to the monitor as he begins to move the handle to the left</u> (Figure 6). The blue dot at (1, 1) disappears and the blue dot at (1, 6) appears.*]

In contrast to the previous glance at the panel, Ivan's handle movement is more immediately cotimed with his turning his gaze to the screen. Having overshot and inspected the panel, Ivan may now be attempting to correct this. He again overshoots his goal in the other direction. It may be that both siblings are now aware of a need to fine tune the precision of the attainment of point (1, 1).

7. *I*: <u>Okay.</u> (Figure 7) [*Ivan switches directions, moving his handle a little bit to the right and continuing to watch the screen. Kayla continues to move her handle to the left.*]

In Line 8 Ivan is possibly again correcting for his own overshooting, or it may be that he is aware that the cursor needs to move to the right and anticipates that perhaps moving his handle to the right will accomplish this. In this sense, the process of perceptuomotor integration includes the development of a practical understanding of those aspects of perceptual experience that *cannot* be integrated with one's own motor actions. Perceptuomotor learning requires sorting integrable and nonintegrable aspects of perceptual and motor experience. We suggest that, in the context of becoming fluent with Drawing in Motion, this gradual separation is constitutive of understanding the parametric nature of the curves being drawn.

8. *I*: now um one point [*Kayla moves her handle a little bit to the left then pauses. She begins to move her handle slightly rightwards and looks down towards the button panel. Ivan switches directions again, moving the handle now to the left and pausing intermittently, looking at the screen. Then, as he says, "one point," he turns his gaze down to the number panel.*]

It is likely that in Line 9 Kayla pauses because she has reached the physical limit for leftward movement of her handle.

9. *I*: six (Figure 8). (3.0) [*Ivan shifts his gaze to the monitor. He briefly moves his handle to the right, then switches directions, moving it to the left and watching the screen. Because Kayla is not moving her handle, he creates a vertical segment. In fact, while Ivan is moving his handle leftwards, Kayla leans over to the button panel and presses the clear button midway through Ivan's leftward handle movement.*]

In Line 10 Ivan appears unaffected by Kayla's erasure of their past trajectory. He does not pause, move his gaze, or comment; he just continues his leftward motion while watching the screen. Thus, we suggest that the scope of Ivan's experiential present does not include retention of the trajectory that has been erased; rather, it consists of more immediately emerging patterns of motor action and sensation. This is the first time that Ivan has had the opportunity to experience a relatively extended period of purely vertical displacement on the monitor, cotimed with his handle manipulation. In other words, during this segment, Ivan may be developing an understanding that vertical displacement is integrable with his handle movement.

10. (7.1) [*Kayla begins to move her handle to the right while Ivan continues to move his handle to the left so that the cursor moves diagonally to the point (1, 6). Kayla continues to walk and move her handle to the right, pausing briefly and intermittently. As she moves her handle between the numbers 5 and 6, she shifts her gaze down to the right side of the number panel. She continues to move the handle rightward as she sweeps her gaze leftward along the number panel then back rightward. Without having looked back up, she stops moving her handle (Figure 9) when she has located it at the number 9, suggesting that she looked down at the panel with the intention of using it to organize a goal-directed movement. Meanwhile, after attaining (1, 6), Ivan moves his handle rightward, then leftward, then rightward again, creating a series of peaks and valleys along the horizontally rightward trajectory.*]

Kayla's unidirectional movement combined with Ivan's changes in direction described in line Line 11 may be another important opportunity for the development of a separation between integrable and nonintegrable aspects of perceptual and motor experience. In other words, the unidirectional nature of Kayla's left-to-right movement provides a backdrop against which the bidirectional nature of Ivan's upward and downward movements may be made salient. That Ivan's handle movements affect vertical displacement but not left-to-right horizontal displacement may be enabling him to refine his practical distinctions between those visual features that are and are not controlled by his handle movements. This highlights the fact that perceptuomotor integration—as a gradual transformation in Ivan's bodily experience with Drawing in Motion—is inextricable from sociality. Ivan's experience of verticality associated with his handle movements is intertwined with his interactions with his sister.

11. (1.6) [*Kayla turns her gaze to the screen, keeping her hands on the handle but not moving it. Ivan watches the screen and produces a small, slow, leftward movement of the handle so that the cursor moves gradually vertically to attain the point (9, 6) with high precision. It takes him about 1.5 seconds to move the cursor less than a single linear unit.*]

We suggest that emerging perceptuomotor fluency is also characterized by the integration of perceptual and motor experience on finer and finer scales. Ivan's slow, precise approach suggests that increasingly small motor acts are becoming integrated with increasingly small changes perceived in vertical displacement.

12. (3.2) [*Ivan moves his handle to the right relatively quickly. He doesn't switch directions or look towards the panel. A little after Ivan begins to move his handle, Kayla begins to move her handle to the left, pausing intermittently but without switching directions. During the motion, she briefly glances down to the panel and back up to the screen (Figure 10).*]

Ivan's motor (moving the handle) and perceptual (seeing the graphical trajectory on the screen) activities are now synchronous. In Line 13 one possibility is that Kayla has noticed that the difference between the point (9, 6) and the point (9, 1) is that the *6* has changed to a *1*, prompting her to move her handle to the *1* location on her panel.

13. I: Kayla, you're **going the** wrong way (Figure 11). [*Ivan moves his handle to the right and watches the screen.*]

K: **what** [*Kayla turns her gaze to Ivan*.]

In Line 14 Ivan is synchronously moving his handle and watching the screen while verbally attributing a particular aspect of his perceptual experience to Kayla's actions. This suggests that he is developing an important aspect of the separation between integrable and nonintegrable perceptual and motor experiences; he has an emerging practical understanding that he cannot control perceived horizontal movement on the graph with the motor action of manipulating his handle. Furthermore, we suggest that, drawing on a wealth of background expectations, he has come to expect that the horizontal aspect of the graphical trajectory over which he has no control must be controlled by Kayla. In other words, rather than attributing the horizontal trajectory to some external factor—for example, some preprogrammed aspect of the instrument—he assumes that it must fall within the domain of Kayla's motor control.

Ivan's appraisal of his sister's movements also underscores the social consequentiality of instrumental fluency. As the motoric and perceptual aspects of Ivan's tool use merge, he simultaneously gains a sense of the relations between his sister's movements and changes on the screen. This emergent understanding in turn enables him to articulate an assessment of his sister's participation.

14. (0.6) [*Ivan moves his handle slightly to the left to fine tune the precision of the cursor's vertical displacement. Kayla turns her gaze back to the monitor but continues to pause her handle movement.*]

15. *K*: no (0.1) way. [*Kayla exhales powerfully, as though laughing.*]

16. *I*: 'kay go

17. (2.0) [*Kayla continues to move her handle slowly to the left. Ivan glances at Kayla then back at the screen, continuing to pause his handle movement. Kayla then glances at Ivan, continuing to move her handle to the left.*]

18. *I*: nope the other way the other way the other way the other way. [*Kayla turns her gaze back to the monitor and continues to move left during the first half of Ivan's verbal utterance. She then looks down toward the panel and begins to move the handle slowly to the right. She looks back up to the monitor and makes a large continuous movement of the handle to the right.*]

The interpretation of the meaning of Ivan's utterances, "you're going the wrong way," and "go the other way" in Lines 14 and 19 seems nontrivial for Kayla. Her glances at the panel several times before and during her leftward movement, the quick speed of her leftward movement, and her laughing exclamation, "no way," in Line 16 suggest that Kayla's leftward movement was confident and goal-directed. It is likely that Ivan's persistent repetition of "the other way" prompts Kayla to entertain several alternatives for what "the other way" might mean.

19. (2.0) [*Kayla moves her handle to the right. The cursor attains the point (9, 1) with high precision. As she approaches the target point, her handle movement slows and she pauses intermittently.*]

Although this analysis has focused more on Ivan, it is important to note that Kayla also exhibits behaviors characteristic of emerging perceptuomotor integration. Specifically, in Line 20, as the cursor approaches the point (9, 1) Kayla continues to watch the screen but significantly slows her handle movement, pausing and inching to the right intermittently. This suggests that she is also integrating perceptual and motor aspects at increasingly refined scales, and in a way that is inextricable from her brother's verbal imperatives.

20. *I*: okay (0.2) now g' the other way

21. (3.2) [*Kayla watches the screen and makes one continuous leftward movement of the handle to complete the challenge (Figure 12).*]

Discussion. Several aspects of perceptuomotor learning are revealed by the details of physical exhibit manipulations, eye gaze, and sparingly produced speech in this episode. We can now envision perceptuomotor learning as a process by which perceptual and motor aspects that are initially associable with either the just-past or about-to become intermingled to the extent that both protentions and retentions are experienced as simultaneously motor and perceptual. In the data, this process was revealed by handle manipulations and glances at the monitor that, initially temporally separated, became increasingly synchronous. In Line 20, for instance, Kayla simultaneously looks at the screen while moving the handle: Her action is *continuously* fully perceptual and motor.

We propose that symbols, such as the graphs drawn on the computer screen, become meaningful through these types of experiences conducive to perceptuomotor integration. For instance, for Kayla, viewing leftward horizontal movement became part of a complex pattern of action that encompassed leaning, pushing, and walking toward her left side; moving within a certain lived range of speed and strength; and slowing down near the destination point. Moreover, this experiential pattern of bodily activity is irreducible to a definitional statement. For example, one can hear or read the definition, "leftward horizontal movement corresponds to moving the *x*-axis handle to the left" without feeling a kinesthetic sense of what kind of bodily poise, effort, and posture is entailed. One's ability to awaken involuntarily kinesthetic echoes in the act of perceiving or creating a symbolic expression could be as central in mathematics as imagining sounding notes can be for a piano player in the act of reading or producing a musical score.

The process by which perceptual and motor aspects become intertwined is highly selective, involving the gradual separation of integrable from nonintegrable aspects of experience. Although Ivan's handle movements during the beginning of the episode suggested a partial expectation that he could affect both vertical and horizontal displacement, his verbal utterances at the end of the episode suggest that he had come to integrate only vertical displacement with his handle movement. Perceptuomotor learning may also be further characterized by the integration of perceptual and motor experiences on finer and finer scales. In other words, Ivan and Kayla did not merely come to associate their handle movements with either vertical or horizontal displacement; their expertise further developed as they came to integrate relatively small handle movements with relatively small displacements, revealed by their gradual approaches to targeted points.

Goodwin (2007) illustrates that common gestures can be *environmentally coupled*, incorporating surrounding objects, attributes, and locations. The same can be noted of conversations: Participants' utterances often incorporate their surroundings in ways that not only make verbal productions highly indexical (i.e., saying "the book" while pointing to a book nearby, might amount to saying something like "the book that I am reading which is about a young poet and is right now on top of the bed") but also can make verbalizations unnecessary. We propose that the indexicality of conversational interaction depends on the relationship between the physical environment and the conversation topic. For example, that Ivan and Kayla's environment incorporated Drawing in Motion may have allowed them to communicate despite sparse verbal production. Even the most striking and forceful verbal expression that occurred in the episode, “nope the other way the other way the other way the other way” (Line 21), was highly indexical. Each time Ivan said, “the other way,” prompted Kayla, we think, to attempt multiple interpretations for what “the other way” could be (e.g., a matter of direction on the graph? or of a change of pace? or stopping?). Furthermore, these interpretations amounted to practical actions (e.g., stopping, shifting gaze) with Drawing in Motion. We suggest, then, that the use of mathematical instruments expands the realm of indexicality accessible to the conversants in a way that enables both highly indexical, environmentally coupled utterances as well as rich, nonverbal communication.

If the surroundings of the conversants exclude items related to the theme of the conversation, utterances may still become environmentally coupled in an imaginary sense. In other words, the participants imagine that they are in another place with entities relevant to their talk, positioned around them, and which can then be pointed at, grabbed, moved, and so forth. This will be the case with the next episode.

Perceptuomotor Integration and Cooperative Imaginative Enactment: Rafaela and Nicolas

This episode occurred toward the end of Rafaela and Nicolas’s (twins, age 12) interview after they had worked on Drawing In Motion. Prior to this episode, the interviewer had asked them questions about their experience with the exhibit and they had reviewed a video screen-capture of the images they produced with the exhibit. At the beginning of the episode, the interviewer positioned his forearm in front of the monitor so that it formed a 45º angle over the axes displayed on the computer monitor (Figure 13). The interviewer asked, “So suppose that you want to do a line that is like this (holds arm at 45 degree angle over grid on monitor). When you were the vertical (points at Nicolas) and you were the horizontal (points at Rafaela), how would you do it?" Rafaela and Nicolas had not produced such a line while working with the exhibit.

This episode provides a window into how Rafaela and Nicolas experienced slope and linearity through Drawing in Motion. Through their imagined enactment of generating the line specified by the interviewer, we see that this experience reflects rich perceptuomotor integration, fluency with the exhibit as a mathematical instrument, and the necessity of cooperative embodied engagement by the users. In this episode, we explore how fluency with a mathematical instrument enables one to imagine, improvise, and enact the generation of novel images even in the absence of the exhibit. Furthermore, the collaborative nature of Drawing in Motion implies that one must also develop a sensitivity to the other user's contributions. With sufficiently attuned sensitivity to the other user's contributions, mathematical objects such as lines with particular slopes, circles, and countless others may be created using the exhibit. A verbal transcription of the episode follows.

N: I would be, since I would be at the top at ten I would have to go down **at the same speed (0.3) she'd** have to go

R: **and I'd have to go**

(0.8)

N: well she'd be at one, so she'd have to go down

(0.4)

N: so she'd go up to 10 and I'd go down to the 0 [*raises pitch*]

(0.5)

Int: so if she goes from (0.1) zero to **ten** [*raises pitch*]

R: **yeah** but we'd

R: **be going at the same time**

N: **so she'd go like this but** I'd go like

N: well, we would both go like that

Annotated episode.

1. *N*: I would be, since I would be at the top (Figure 14) [*Nicolas moves his right hand upward and to his anterior.*]

Nicolas is setting the stage for enacting what he would have to do with the handle to produce the line. In this preparation, Nicolas uses his right arm to establish his beginning position within the grid: He raises his right arm in front of his body to indicate the position of the cursor in the upper-left portion of the grid, reflecting his role of controlling the movement along the *y*-axis. Together, Nicolas's verbal utterance, "I would be at the top at 10," and his closed hand gesture suggest that Nicolas imagines himself occupying this location within the grid on the monitor.

2. *N*: at 10

Nicolas is transitioning from marking the initial vertical location line on the grid to a more direct enactment of exhibit manipulation by gripping the handle. Note how being at *10* is not merely a matter of number-location correspondence or of establishing a Cartesian connection. In Lines 1 and 2 Nicolas is pointing out *his* being at the top: over there, where the *10* is. Being at the top entailed being bodily at the left side of handle's track *and* on the upper side of the computer screen, *while* getting ready to start drawing the desired line.

3. *N*: I would have [*Nicolas shifts his right hand a bit to the left.*] to go down (Figure 15) [*Nicolas moves his hand to the right, keeping his fingers curled inwards*.]

Nicolas completes the preparation for the enactment of the movement of the handle by moving his closed fist more to the left of his body. This movement places the handle all the way to the left on the *y*-axis panel, at the initial position of 10. He is now ready to enact how he would move the handle. As he says, "down," he moves his hand to the right, as though pulling the handle. This action of moving the handle to the right on the exhibit results in decreasing the value of the *y*-coordinate on the monitor, correctly changing the value of the *y*-coordinate to create the desired line. The simultaneity of his right hand moving to the right and vocally uttering "down" reveals his fluent perceptuomotor integration: He is moving his hand to the right while envisioning the cursor moving down on the screen. Nicolas is acting a particular symbol use: Vertical movement down is recognized as a hand displacement to the right.

4. *N*: **at the same speed (0.3)** [*Nicolas keeps his right hand in the handle-gripping shape.*]
   R: **and I'd have to** [*Rafaela lifts her left hand up towards her right shoulder, curls her fingers inwards, then drags her left hand to the left across the front of her body.*]

Note that Drawing in Motion does not require the users to move the handles at a constant speed throughout the trajectory to produce a straight line. However, there are different possible perceptions that could motivate Nicolas to state that he would have to go "at the same speed." It could be that he sees that the same relative speeds are required to keep the line straight or it could be that keeping each speed constant helps to coordinate with his sister. If he goes at a constant speed his sister can do so as well, which is the easiest condition to enact jointly.

5. *R*: **go** [*Rafaela relaxes her left hand back to her lap.*]
   N: **she'd** have to go (0.8) [*Nicolas continues to hold the shape in his right hand as though it is gripping the handle.*]

6. *N*: well <u>she'd</u> (Figure 16) [*Nicolas extends his left index finger in front of him. He maintains the shape in his right hand, but draws it in a bit towards his body.*]

Right after Rafaela starts to describe how she would have to move, Nicolas continues to dominate the conversation as he enacts their joint action: With his left hand he begins to enact Rafaela's role in creating the line. Note that as his left hand moves to enact Rafaela's motion, he maintains the tonicity (i.e., level of muscular engagement) of his right hand that had enacted his movements with the handle. We suggest that this utterance reveals the multiplicity of Nicolas's body; his left hand enacts Rafaela's motions and his right hand is temporarily quiet, but not relaxed, preserving the sense of his simultaneous motion on the *y*-axis handle. This multiplicity seems to sever the original temporality, juxtaposing two events that would occur simultaneously on the exhibit. This illustrates a very important sense in which imagining is not a kind of realistic simulation, but a transformation of actual events that breaks their original spatiotemporal character by juxtaposing aspects that were spatially distant (e.g., the motion of the handle and the graphical shape) and making simultaneous events sequential and sequential events simultaneous.

The multiplicity of *x*-handle and *y*-handle movements enacted through left and right hands also again foregrounds the idea that the distinction between individual and interpersonal processes does not inherently characterize experience. Over the course of a series of bimanual gestures throughout this episode, Nicolas imagines a coordinated, multiparty activity with his sister through the movements of his own body.

As he did for his role, Nicolas is beginning to set the stage for the enactment of his sister's role. He uses his left hand to point to the beginning location of the handle for his sister on the *x*-axis at 1, but there is a subtle difference in this preparation for his sister's manipulation. His pointing to the location of the *1* signals that he is referring to an activity that would occur outside of his peripersonal space by someone other than himself; in Line 3 he kept his hand closed when he indicated his starting position on the monitor, situating his activity within his peripersonal space as it would be if he were standing at the exhibit.

7. *N*: be at one [*Nicolas extends his left finger toward the computer monitor.*] so she'd have to go down (Figure 17) (0.4) [*Nicolas traces the diagonal line in the air with his left index finger, moving from the top left to the bottom right.*]

The statement, "so she'd have to go down," indicates that Rafaela would, in one form or another, be going "down" even though she would increase the numbers on the horizontal scale (from 1 to 10) and move the handle horizontally. We suggest that this "down" means that she would move the handle down the track to the right side, away from Nicolas's body, which is a sensible point of view, given that Rafaela would be walking in his extrapersonal space. This interpretation is consistent with how one would refer to a train "going down the tracks" as it pulls out of the train station. Meanwhile, Nicolas maintains the tonicity of his right hand as it marks his role in creating the line that he had enacted just seconds before.

8. *N*: so she'd go [*Nicolas curls his left fingers inwards and moves his left hand slightly to the left*] up to ten (Figure 18) [*Nicolas moves his left hand to the right, across the front of his body. He lifts his right hand upward.*]

Imagining what his sister would have to do, Nicolas prepares his left hand to enact the movement his sister would make with her handle. With his hand closed in a fist, Nicolas moves his arm to the left as if he were standing in front of the *x*-panel on the exhibit, grasping the handle at the *1* position. To maintain his right hand as if it were holding his handle, Nicolas lowers his right hand to allow for the movement of his left hand as it enacts the movement of Rafaela's handle. His lifting of the right hand is likely to be an expression of his anticipation that he will need free room to displace his left arm to the right. This illustrates how in imagining, some of the actions respond to "logistic" needs (i.e., it was not that he was moving the *y*-axis handle up). For Nicolas, moving his sister's handle to the right is the same as going "up to ten."

9. *N*: and I'd [*Nicolas moves his right hand a little to the left*] go down to the zero [*Nicolas raises his pitch as though questioning. He moves his right hand to the right, then pauses his right hand while maintaining the hand shape*.]

Nicolas prepares his right hand to enact the movement he would need to make by moving it slightly to the left. He then enacts the handle movement.

10. (0.5) [*The interviewer turns his gaze to the screen.*]

11. *Int*: so if she goes from (0.1) [*The interviewer uses his right index finger to trace along the* x-*axis displayed on the mnitor, moving his finger from the left to the right endpoint.*] zero to **ten** [*The interviewer raises his pitch.*]

12. *R*: **yeah** but <u>we'd</u> (Figure 19) [*Rafaela <u>lifts her right hand</u> from her lap. The interviewer returns his gaze to Rafaela and Nicolas. Nicolas continues to maintain the tonicity of the handle-holding shape in his right hand.*]

13. *R*: **be going**

N: **so she'd** [*Nicolas moves his right hand to the left.*]

It appears that Nicolas has switched hands—now the right hand is for Rafaela and the left for himself. As he says "so she'd," he moves his right arm to the left with his hand clenched in preparation to enact how Rafaela would move her handle.

14. *R*: **at the same <u>time</u>** (Figure 20) [*Rafaela moves her hand diagonally up and to the right in front of her body.*]

N: **go like this <u>but</u> [***Nicolas moves his right hand to the right across his body. At the same time, he curls his left fingers in and lifts his left hand up.*]

Rafaela's right hand does not appear to trace the line in question, but is rather an enactment of moving at the same time. Rafaela may be enacting the movement of the cursor as it crosses the screen on the monitor as they move the handles (this could possibly be an enactment of how both of them would move the handles at the same time, but the lack of handle-gripping gesture makes this less likely). Rafaela feels the need to emphasize the simultaneity of their contributions as Nicolas has produced them in a temporally linear fashion. This simultaneity is crucial with respect to slope, and her voicing this emphasis provides evidence of the necessary cooperation between users and coordination of quantities for slope in the context of the exhibit. Meanwhile, Nicolas appears to enact Rafaela's movements with his right hand while preparing to enact his own movements with his left hand, again embodying a multiplicity.

15. *N*: I'd go <u>like</u> [*Nicolas maintains the position and shape of his right hand and moves his left hand to the right across the front of his body. Rafaela relaxes her right hand back to her lap.*]

With his left hand clinched, Nicolas moves his left arm to the right, enacting with his left hand what he would need to do with the handle.

16. *N*: <u>well</u> (Figure 21) [*Nicolas <u>moves both hands to the left</u> in front of his body, keeping both hands in the handle-holding shape.*]

Maintaining his hands clenched and side-by-side, Nicolas moves both hands to the left to prepare to enact how he and his sister would move the handles simultaneously.

17. *N*: well, we would both go like <u>that</u> (Figure 22) [*Nicolas moves both hands to the right, keeping the fingers curled in as though each hand is grasping a handle.*]

This utterance again reveals the multiplicity of the body. Nicolas uses one hand to enact his movement and the other his sister's, but he is no longer breaking up temporality. The bodily multiplicity now functions to reflect directly the cooperation and simultaneity of their handle movements. He moves both hands to the right as if he were moving both handles on the exhibit.

Discussion. In this episode, Rafaela and Nicolas imaginatively enacted the production of a negatively sloped line, a line that they had not actually produced using Drawing in Motion. That they succeeded in explaining the production of a novel line suggests that their learning was not restricted to a recollection and reproduction of previous cases. Rather, their emergent fluency with Drawing in Motion empowered them to grapple with a range of both familiar and unfamiliar cases. We see the origin of this imaginative flexibility in that they had engaged in *bodily orientations* towards the exhibit. That is, Rafaela and Nicolás had used the exhibit by enacting physical arrangements that were grounded not only in the coordinates of their bodies (e.g., above me, below me, to my right, to my left) but also in felt or lived qualities that made available to them a nuanced situational sensitivity to relative positions, velocities, and so on. In general, we orient ourselves toward a range of possibilities: There are many speeds that count as "going faster," or many angles that encompass "turning right," with these ranges being expanded or contracted by the circumstances. These ranges of possibilities are delimited by unspecified boundaries. For instance, it is likely that Nicolas's and Rafaela's line-drawing experiences were assuming constant individual speeds, which might have been expressed by Nicolas in Line 4 ("at the same speed"). If this were the case, the constancy of individual speeds would be an unspecified boundary for their expectations regarding line drawing. Fluency with bodily oriented enactments, we propose, allowed Nicolas to perceive a negatively sloped line as both him and his sister moving their handles to their right sides, or Rafaela to clarify that their body motions had to be simultaneous. We propose that Nicolas's and Rafaela's perceiving the line represented by the interviewer's arm entailed their enactment of bodily orientations with which they had become fluent. In Lines 1 and 2, for instance, Nicolas responded to perceiving the line by orienting himself as "being at the top" and at "10."

That Rafaela and Nicolás achieved a level of fluency with Drawing in Motion is apparent in the details of their imaginative enactments during this episode. For example, in Line 2 perceptuomotor integration is evinced in the ease with which Nicolas associates locations in the grid on the monitor with corresponding handle positions. Importantly, the activity in this episode reveals that the nature of imaginative enactments enabled by perceptuomotor integration is not a literal reproduction of the actions that would take place in using the exhibit. Many cognitive scientists currently operate within a simulation theory of motor imagery that holds that imagined actions are relatively covert stages of actual motor activity (Jeannerod, 2001). Accordingly, some gesture researchers argue that gestures are produced by perceptuomotor simulation (Hostetter & Alibali, 2008). Although such accounts emphasize the continuity between the actual and the imagined, the events in this episode reveal how present imaginative enactments constitute a nontrivial transformation of actual motor actions. For example, in Lines 6–8 Nicolas began to enact Rafaela's actions with his left hand, while pausing but still holding with his right hand the other imaginary handle. In actuality, the two handles of Drawing in Motion cannot be operated simultaneously by a single user. Furthermore, actual production of the negatively sloped line requires simultaneous movement of the two handles, but Nicolas initially enacted these sequentially. In other words, we submit that imagining is not a replication but a *transformation* of the actual that (a) may involve making sequential the simultaneous or simultaneous the sequential and (b) may bring distant elements into proximity or separate neighboring elements.

Such transformations may also overlap or fuse the attributes of signified and signifier. For example, in Line 9, Nicolas said, "I'd go down to the zero," while moving his hand to the right. In this utterance Nicolas accompanied the word *down* with a hand movement to the right, the former describing a feature of the graphical signifier (i.e., the line going down on the graphical display) and the latter enacting a quality of the signified action (i.e., the *y*-axis handle being moved to his right side). Nemirovsky and Ferrara (2009) described these features of the imaginary using the word *juxtaposition*, that is, by the juxtaposition of items or actions that were originally (i.e., in actuality) separated temporally, spatially, and thematically.

An important aspect of understanding slope entails the ability to perceive it as a ratio of *two* independent variables, which can be expressed by the steepness of a single line or a number along a single scale. The experience with Drawing in Motion elicits this duality by making salient the need to modulate one's actions in response to the actions of the other. What counts for the graphical production is not the activity of each participant in isolation, but the coordination between them. The downward line with which Nicolas and Rafaela were grappling in this episode constrains relative speeds while leaving individual speeds underspecified. In other words, gaining fluency with Drawing in Motion entails the incorporation of both users' actions so that the perception of a graphical shape comes to elicit the motor activity of both participants. Evidence of such "mirror" perceptuomotor integrations include Rafaela's assertion that their motions would be going "at the same time" (Line 14) or Nicolas's conclusion that "we would both go like that" (Line 17).

The observations that the performance with Drawing in Motion is collective and that subsequent imaginary enactments involve the juxtaposition of multiple heterogeneous elements calls attention to the multiplicity afforded by the body. That is, we suggest that the unity of the body is not a given sustained by its anatomic and physiological cohesiveness. Rather, the unity of the body emerges out of a complex multiplicity (Mol, 2002), in which one arm can be "me" while the other one is "you," or my fingers can touch something present while my words locate it in a distant past. Nicolas imaginatively encompassed his and his sister's joint activity with his own body by enacting parallel streams of activity that would occur simultaneously if he and his sister were actually carrying out the activity on the instrument. This multiplicity, which he preserved throughout the enactment (for example, by maintaining the tonicity in an inactive hand as if he were still grasping one handle while moving the other), allowed Nicolas to provide a detailed enactment of each contribution sequentially while still emphasizing the cooperative simultaneity necessary to successfully complete the task.

Discussion

This study contributes to embodied theories of mathematical thinking and learning by illustrating the beginnings of a theoretical framework for understanding the nature of mathematical tool fluency in terms of transformations of lived bodily experience. Through a detailed analysis of two episodes involving emerging fluency with a mathematics exhibit in a science center, we characterized mathematics learning as a process in which the perceptual and motoric aspects of using a mathematical instrument become intertwined. By describing emergent mathematical tool fluency in terms of perceptuomotor integration, we purposefully eschewed accounts of mathematics learning that appeal to hypothetical mental structures or categories of reasoning that are divorced from the irreducible perceptuomotor richness of mathematical experience. In order to foreground this nondualist approach we adopted the language of *mathematical instruments*, and conceptualized mathematical instruments analogously to musical instruments: Fluency with a mathematical instrument enables "doing" certain kinds of mathematics, just like fluency with a musical instrument can be a crucial skill for certain forms of music playing and for one's membership in sociocultural communities that value and preserve such musical practices.

Each of the episodes analyzed in this study served to illustrate how perceptuomotor integration can be the basis of a type of mathematical learning in which we generalize by means of enacting bodily orientations and give meaning to symbols by awakening perceptual and motoric patterns. The first episode offered an example of perceptuomotor learning in terms of the mergence of perceptual and motoric aspects of using Drawing in Motion. Through a microanalysis of Ivan and Kayla's attempt at Challenge 1 we found that (a) perceptuomotor learning is highly selective, involving the gradual separation of integrable from nonintegrable aspects of experience, and (b) symbols become meaningful through experiences conducive to perceptuomotor integration that are irreducible to definitional statements. A close analysis of the second episode revealed how emergent instrumental fluency enables imaginative approaches to novel mathematical questions that are grounded in felt qualities that make available to participants a nuanced situational sensitivity to relative positions, velocities, and so on. These imaginative enactments are not mere re-evocations of the past but are marked by profound transformations of prior experiences.

The analysis of the selected episodes includes diverse examples of how our emergent framework for understanding the development of fluency with a mathematical instrument for the parameterization of time differs from prevalent trends in the mathematics education literature on the understanding of functions. Rather than examining the episodes with an eye toward identifying the participants' mental schemes, forms of reasoning, cognitive structures, procedural strategies, and the like, we strived to grasp their lived experiences, in other words, the temporal flows of perceptuomotor activities they inhabited bodily, emotionally, and interpersonally. For instance, instead of interpreting Nicolas's utterance, "I would be at the top at 10" (Lines 1 and 2), as the mental possession of some aspect of a Cartesian connection, we worked to articulate how it was part of his *being at a plac*e that was simultaneously in front of the handle, on the computer screen, in readiness to move right and down, and so on. Similarly, rather than proposing some specific covariational reasoning "behind" or "underneath" Rafaela's insistence that "we'd be going at the same time" (Lines 12–14) or Nicolas's conclusion that "we would both go like that" (Line 17), we attempted to trace the roots of these utterances in their local history and their ongoing efforts to imagine the use of an absent instrument. Some readers might object that this focus on the phenomenology of lived experience will prevent us from reaching any level of generality beyond the short duration of these episodes: We'd be bound to stay within the actuality of their actions and utterances irrelevant for what others, or themselves at another time, would say or do. We hope to have shown that this is not the case. By tracing the particulars of their imaginative work, we endeavored to advance general theses on the roles of perceptuomotor integration with Drawing in Motion for the open-ended use of position versus position graphs obtained by parameterizing time, the transformative juxtaposition of elements to solve new problems, such as the tracing of a negatively sloped line, or the enactment of kinesthetic memories in the interpretation of symbols.

Although this study is only a modest contribution to a larger agenda of resituating mathematics learning in lived bodily experience, we feel the beginning framework and microanalytic results sketched in this article have several important implications for the fields of mathematics education and informal learning, and for their intersection. First, the results of our analyses reopen questions related to the temporal scale of learning. It is common for educational researchers to naturalize the assumption that learning is a long-term affair; this way of thinking is perhaps most poignant within informal education literatures that express anxiety about attributing learning to the seemingly ephemeral moments that constitute a museum visit (e.g., Rennie & Johnston, 2004). Yet our analyses prompt us to invite the reader to consider that learning is a complex phenomenon that may be occurring on multiple time scales. Without diminishing the value of understanding learning on time scales at the level of hours, days, years, or decades, the microanalytic approach taken in this study also suggests that, when learning is understood in terms of nuanced shifts in bodily experience, it is also possible to investigate learning processes at the level of seconds or minutes. The first episode demonstrated a noticeable shift in Ivan's and Kayla's bodily engagement over the course of less than a minute. More generally, although none of the visitors to Drawing in Motion spent more than 25 minutes engaged with the exhibit, during subsequent interviews they consistently provided nuanced reenactments of previous graphical productions, imagined approaches to novel mathematical tasks requiring the use of the instrument, and improvised on-the-spot solutions to nontrivial problems requiring coordination of changes in x- and y-values. The second episode exemplifies this.

Second, our careful attention to the nuanced participation of learner engagement and communication points to an important limitation of the discourse analytic research both within mathematics education and informal learning studies. Over the past decade or so there has been a rapid proliferation of both school-based and museum-based studies that examine the details of learner discourse. For example, Ryve (2011) reviewed 108 mathematics education research studies that employed some form of discourse analysis, and researchers in the area of visitor studies have been using discourse analysis to understand family and school-group learning in museums (DeWitt & Hohenstein, 2010a, 2010b; Leinhardt, Crowley, & Knutson, 2002). The vast majority of these studies, however, only take verbal data into account, ignoring the participation of body motion, gesture, eye gaze, and interaction with the material environment. Yet, a quick read over the verbal portions of the transcripts for the two episodes analyzed in the present study attests that neither of these episodes would have been intelligible using a verbal-dominated analysis approach. Instead, the development and imaginative use of instrumental fluency inheres in a complex landscape of multimodal expression that includes, but is not reducible to, speech.

An analytic lens that includes multiple modalities, combined with the nondualist approach to tool use sketched in this study, points to a third important implication for mathematics education research and research on learning in informal settings. Although many mathematics and informal education researchers both (a) value engaging learners' bodies through hands-on or interactive activities or exhibits and (b) acknowledge a general appreciation for the contributions of perception and action to learning, we believe that even this educational literature often continues to naturalize dichotomies between thinking and acting, perceiving and conceiving, and mind and body. For example, we agree with Farnell (1996) that this is the case for metaphor-based theories of embodiment that relegate bodily experience to a metaphorical source-ground—rather than genuine constituent of—learning and knowing (cf. Lakoff, 1993; Lakoff & Núñez, 2000). Similarly, within informal education, researchers and practitioners at once laud hands-on engagement while simultaneously questioning its epistemic status; for instance, informal educators often add the term *minds-on* to discussions of hands-on learning (National Research Council, 2009) or write about *kinesthetic-cognitive conflict* (Plummer, 2009). Working from a nondualist approach, we suggest that, in the contexts of both school-based and informal mathematics education, researchers replace anxiety about the extent to which bodily activity should count as mathematics with analytic practices that seek to uncover the nuanced ways in which multimodal engagement is genuinely constitutive of knowing.

With respect to the existing research on the teaching and learning of functions, this article suggests the pedagogical importance of the parameterization of time, and the possibility that, if provided with suitable mathematical instruments and practices, learners from an early age can engage with them in rich and nuanced ways. As an early milestone toward a nondualist approach to embodied mathematics learning, this study opens many questions and suggests numerous directions for further research. For example, although this study investigated one type of mathematical instrument, we conjecture that fluency with a wide variety of tools, technologies, and artifacts can be described in terms of perceptuomotor integration. In each case, the detailed nature of this process—including, for example, which perceptual and motoric elements become intertwined—could be investigated, along with the kinds of mathematical understandings entailed by this emergent fluency. For example, although Drawing in Motion allows for the exploration of certain parametric functions, future research could explore the suitability of other mathematical instruments to learning different areas of mathematics.

The visitors who used Drawing in Motion became acquainted with some aspects involved in the understanding of slope, but not with others, such as with its formal-symbolic algebraic expressions and definitions. Could Drawing in Motion be part of a larger set of exhibits to elicit these other aspects? Moreover, what would count as a mathematical instrument in the case of people using only mathematical symbols written on paper or on a blackboard? What if the use of Drawing in Motion were part of a sequence of activities encompassing sessions of formal or classroom education? It is well known that maintaining fluency with instruments often requires constant practice. What happens if one does not use a mathematical instrument for some years? Does the corresponding imaginative flexibility fade out? How do diverse social interactions shape what is learned with a mathematical instrument? In what ways does fluency with a mathematical instrument turn out to be relevant to becoming a member of certain mathematical communities?

We hope that this article will contribute toward the advancement of mathematical embodied cognition, a growing appreciation for the richness of microethnography in understanding mathematical thinking and learning, and the increasing recognition of informal mathematics education.

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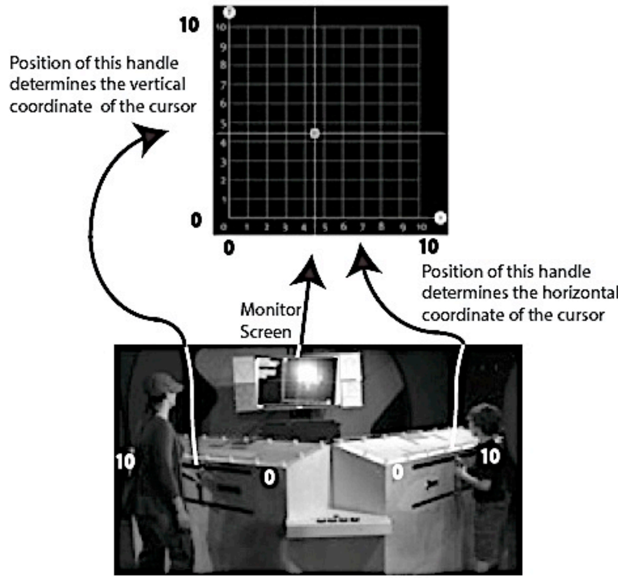

Figure 1: Image and diagram of Drawing in Motion.

Figure 2. Ivan says “move here” and looks at the panel. Kayla looks at the screen.

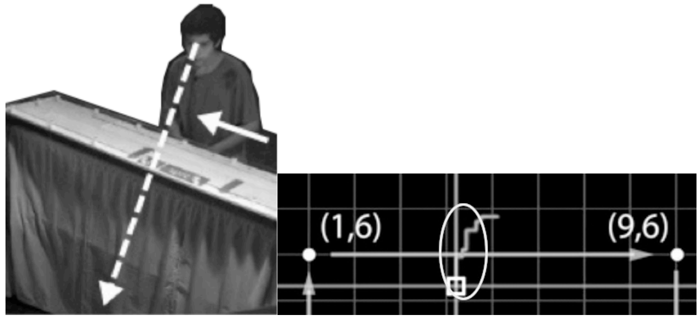

Figure 3. Ivan moves his handle to the right, then looks to the screen.

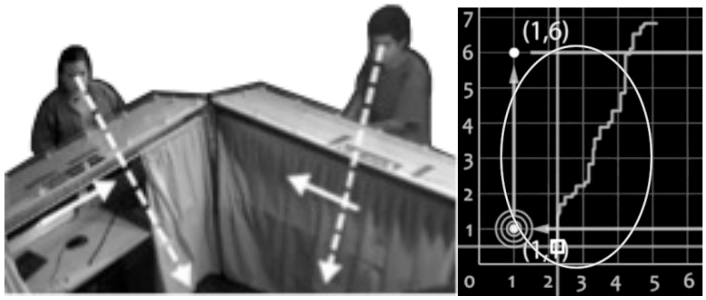

Figure 4. Ivan moves his handle right, and Kayla moves her handle left. They look at the screen.

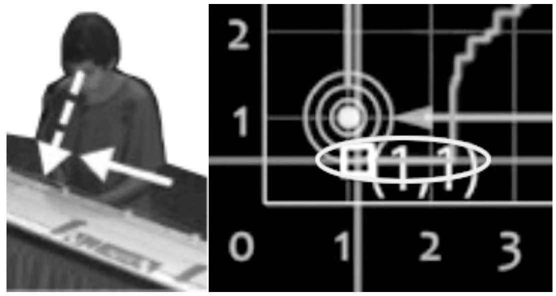

Figure 5. Ivan looks at the panel. Kayla produces a horizontal segment.

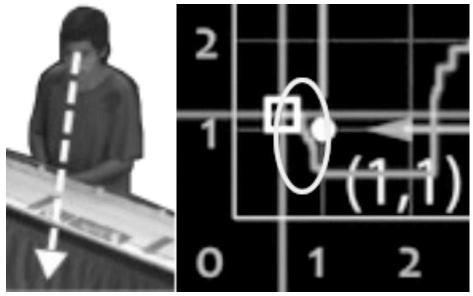

Figure 6. Ivan and Kayla make small handle movements. Ivan looks at the screen.

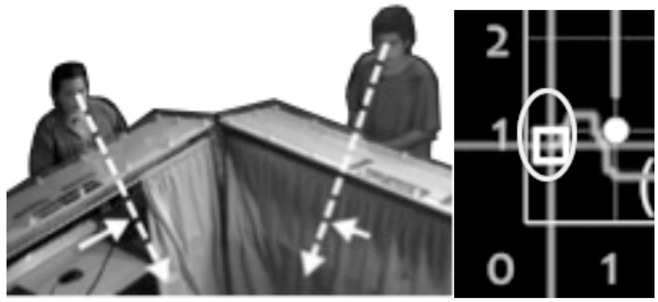

Figure 7. Ivan says “okay” and switches directions.

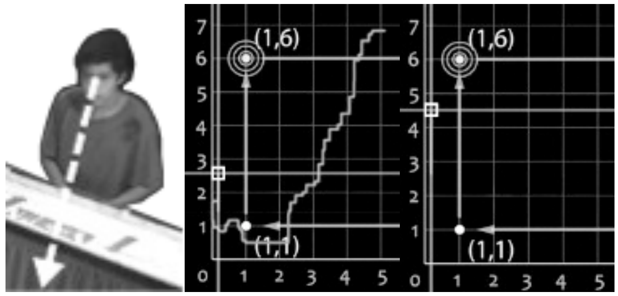

Figure 8. Ivan moves the handle left. Kayla presses the clear button.

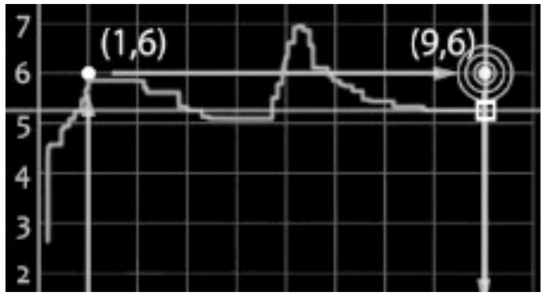

Figure 9. Kayla moves the handle right. Ivan moves the handle right, then left, then right.

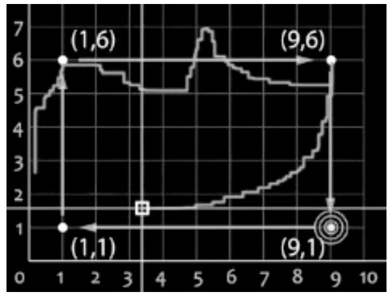

Figure 10. Ivan moves the handle right. Kayla moves the handle left.

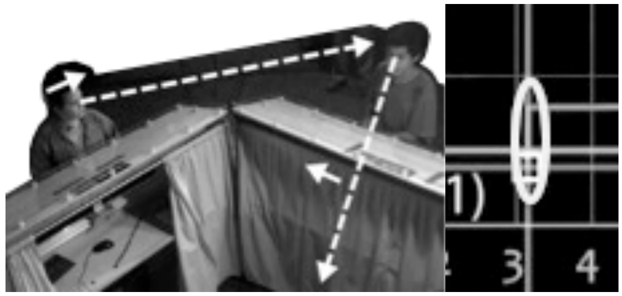

Figure 11. Ivan says, “you’re going the wrong way.”

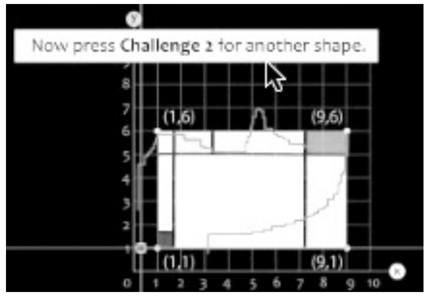

Figure 12. Final graphical display.

Figure 13. The interviewer gestures the line segment.

Figure 14. Nicolas says, “I would be at the top.”

Figure 15. Nicolas says, “I would have to go down.”

Figure 16. Nicolas says, “well she’d.”

Figure 17. Nicolas says, “she’d have to go down.”

Figure 18. Nicolas says, “so she’d go up to 10.”

Figure 19. Rafaela says, “yeah, but we’d.”

Figure 20. Rafaela says, “at the same time,” while Nicolas says, “go like this but.”

Figure 21. Nicolas says, “well,” and moves both hands to the left.

Figure 22. Nicolas says, “well, we would both go like that.”